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IDENTIFICATION OF LINEAR SYSTEMS WITHOUT ASSUMING STABILITY AND MINIMUM PHASE*

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ABSTRACT

For both discrete- and continuous-time linear input-output systems, consistent estimates are given for unknown coefficients, system orders and time-delay. The proposed methods are characterized by the fact that there is no requirement for stability, minimum phase and any other behaviour of the system and by the fact that the designed experiment is diminishing. The latter fact is important when adaptive control problem is simultaneously solved.

Keywords: identification, parameter estimation, continuous-time system

I. INTRODUCTION

We consider the identification problem for unknown orders, time-delay and matrix coefficients of the linear systems which are described as

$$A(z)y_n = B(z)u_n, \ n \geqslant 0 \text{ and } y_n = 0, \ u_n = 0 \text{ for } n < 0 \tag{1}$$

in the discrete-time case, and as

$$A(s)y_t = B(s)u_t, \ t \geqslant 0 \text{ and } y_t = 0, \ u_t = 0 \text{ for } t < 0 \tag{2}$$

in the continuous-time case, where z and s are the shift-back operator $(zy_n = y_{n-1})$ and the integral operator $\left(sy_t = \int_0^t y_\lambda d_\lambda \right)$ respectively, y_n denotes the m-output and u_n denotes the l-input and $A(z)$, $B(z)$ refer to the matrix polynomials

$$A(z) = I + A_1 z + \cdots + A_{p_0} z^{p_0}, \ p_0 \geqslant 0, \tag{3}$$

$$B(z) = B_{d_0} z^{d_0} + \cdots + B_{q_0} z^{q_0}, \ q_0 \geqslant d_0 \geqslant 1, \tag{4}$$

whose orders, time-delay and coefficients are unknown, i. e., the delay d_0, the system orders (p_0, q_0) and the coefficients

$$\theta^\tau(p_0, d_0, q_0) = [-A_1 \cdots - A_{p_0} B_{d_0} \cdots B_{q_0}] \tag{5}$$

are to be estimated.

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This problem has been treated in various areas: In the time series analysis, this is the ARMA model with $\{u_n\}$, $n \in (-\infty, \infty)$ being an unavailable martingale difference sequence and in estimating its unknown parameters one usually assumes[1-3] that the system is stable or minimum-phase or even both. In the theory of linear systems when identification problem is concerned, the system input $\{u_n\}$ has to be designed (experiment design), the orders and the time-delay of the system are often supposed to be known[4-5] and one of the conditions such as stability condition, minimum-phase condition and the condition consisting in $\|y_n\| + \|u_n\| = O(n^\alpha)$, $\alpha > 0$ is assumed to be satisfied[4-5]. In adaptive control theory, the input is designed with purpose not only for identifying the system but also for controlling the system and the assumptions made on the system are almost the same as those indicated above for theory of linear systems[5-11].

In this paper, not imposing on the systems (1) and (2) any conditions like stability, boundedness or growth rate condition for the system input-output, we design inputs for systems (1) and (2) respectively, and using the least squares algorithms, we obtain consistent estimates for the delay, orders and coefficients of the systems. The convergence rates of estimates for system coefficients are also established. The designed input is diminishing and this is important when adaptive control problem is simultaneously treated.

II. Identification of Discrete-Time Systems

We now consider the systems (1), (3) and (4), where the true time-delay d_0 and the orders (p_0, q_0) of the system are unknown, but assume that they belong to finite sets. To be specific, we need the following condition H_1:

H_1. There are known integers p^*, q^* and $d^* \in [1, q^*]$ such that

$$(p_0, q_0) \in M \triangleq \{(p, q):\ 0 \leqslant p \leqslant p^*,\ d^* \leqslant q \leqslant q^*\}, \tag{6}$$

$$d_0 \in \overline{M} \triangleq \{d:\ d^* \leqslant d \leqslant q^*\}. \tag{7}$$

This condition means that the upper bounds for p_0 and q_0 are available.

We also need the following identifiability condition H_2:

H_2. $A(z)$ and $B(z)$ have no common left factor and A_{p_0} and B_{q_0} are of row full rank.

Let $\{v_n, \mathfrak{F}_n\}$ be an arbitrary l-dimensional martingale difference sequence with properties

$$Ev_nv_n^\tau = \frac{1}{n^\varepsilon} I,\ \|v_n\|^2 \leqslant \sigma^2/n^\varepsilon,\ \varepsilon \in \left[0, \frac{1}{2(t_0+1)}\right),$$

where $t_0 \triangleq (m+1)p^* + q^*$, $\sigma^2 > 0$ is a constant and $\mathfrak{F}_n \triangleq \sigma\{v_i, i \leqslant n\}$. $\{v_n\}$ serves as the excitation source for the system and makes its identification possible.

Further, let u_n^0 be the l-dimensional input, which is $\sigma\{y_i, 0 \leqslant i \leqslant n\}$ measurable and is designed for control purpose. For pure identification we may set $u_n^0 \equiv 0$, while in the case of adaptive control we allow u_n^0 to grow up but not faster than

$$\sum_{j=0}^{n} \|u_j^0\|^2 = O(n^{1+\delta}), \quad \delta \in \left[0, \ \frac{1-2\varepsilon(t_0+1)}{2t_0+3}\right]. \tag{8}$$

Finally, for identifying d_0 and (p_0, q_0), we set the system input

$$u_n = u_n^0 + v_n. \tag{9}$$

We now describe the estimation algorithms: For any $(p, q) \in M$ and $d \in \overline{M}$, set

$$\theta^\tau(p, d, q) = [-A_1 \cdots - A_p B_d \cdots B_q], \tag{10}$$

$$\varphi_n^\tau(p, d, q) = [y_n^\tau \cdots y_{n-p+1}^\tau u_{n-d+1}^\tau \cdots u_{n-q+1}^\tau] \tag{11}$$

with agreement $A_i = 0$ for $i > p_0$ and $B_j = 0$ for $j < d_0$ and $j > q_0$.

Given initial value $\theta_0(p, d, q)$, the least squares estimate $\theta_n(p, d, q)$ for $\theta(p, d, q)$ is defined by

$$\theta_n(p, d, q) = \left(\sum_{i=0}^{n-1} \varphi_i(p, d, q)\varphi_i^\tau(p, d, q) + I\right)^{-1} \sum_{i=0}^{n-1} \varphi_i(p, d, q) y_{i+1}^\tau \tag{12}$$

or equivalently by the recursive algorithm

$$\begin{aligned}\theta_{n+1}(p, d, q) = \theta_n(p, d, q) + b_n(p, d, q)P_n(p, d, q)\varphi_n(p, d, q)\\ \cdot (y_{n+1} - \varphi_n^\tau(p, d, q)\theta_n(p, d, q)),\end{aligned} \tag{13}$$

$$\begin{aligned}P_{n+1}(p, d, q) = P_n(p, d, q) - b_n(p, d, q)P_n(p, d, q)\varphi_n(p, d, q)\\ \cdot \varphi_n^\tau(p, d, q)P_n(p, d, q), \quad P_0 = I,\end{aligned} \tag{14}$$

$$b_n(p, d, q) = (1 + \varphi_n^\tau(p, d, q)P_n(p, d, q)\varphi_n(p, d, q))^{-1}. \tag{15}$$

Let $\{a_n\}$ be any sequence of real numbers satisfying

$$a_n \xrightarrow[n\to\infty]{} \infty \text{ and } a_n/n^{1-(1+t_0)(\varepsilon+\delta)} \xrightarrow[n\to\infty]{} 0. \tag{16}$$

It is worth noting that $1-(1+t_0)(\varepsilon+\delta) \geqslant (1-\varepsilon)/2 > 0$, and hence the existence of $\{a_n\}$ is undoubted.

Set

$$\sigma_n(p, d, q) = \sum_{i=0}^{n-1} \|y_{i+1} - \theta_n^\tau(p, d, q)\varphi_i(p, d, q)\|^2 \tag{17}$$

and

$$CIC_1(p, q)_n = \sigma_n(p, d^*, q) + (p+q)a_n. \tag{18}$$

The estimate (p_n, q_n) for (p_0, q_0) is defined by minimizing $CIC_1(p, q)_n$:

$$(p_n, q_n) = \arg\min_{(p,q)\in M} CIC_1(p, q)_n, \quad n \geqslant 1 \tag{19}$$

and the estimate d_n for d_0 is given by

$$d_n = \arg\min_{d\in\overline{M}} CIC_2(d)_n, \quad n \geqslant 1, \tag{20}$$

where

$$CIC_2(d)_n = \sigma_n(p_n, d, q_n) - d \cdot a_n \tag{21}$$

with p_n, q_n obtained from (19).

Theorem 1. *Assume that Conditions* H_1 *and* H_2 *hold and* the *control* $\{u_n\}$ *is defined by* (9) *with* (8) *satisfied. Then*

$$(p_n, d_n, q_n) \xrightarrow[n\to\infty]{} (p_0, d_0, q_0), \text{ a. s.} \tag{22}$$

and

$$\|\theta_n(p_n \vee p_0, d_n \wedge d_0, q_n \vee q_0) - \theta(p_n \vee p_0, d_n \wedge d_0, q_n \vee q_0)\|$$
$$= O\left(\frac{1}{n^{1-(1+t_0)(\varepsilon+\delta)}}\right), \text{ a. s.}, \tag{23}$$

where $a \wedge b$ *and* $a \vee b$ *mean* $\min(a, b)$ *and* $\max(a, b)$ *respectively.*

This theorem says that (p_n, q_n) and d_n given by (19) and (20) are consistent estimates for system orders and time-delay respectively, while (23) indicates the convergence rates of estimates for system coefficients:

III. Proof of Theorem 1

For every $(p, q) \in M$, denote by $\lambda_{\min}^{(p,q)}(n)$ the smallest eigenvalue of

$$\sum_{j=0}^{n-1} \varphi_i(p, d^*, q)\varphi_i^\tau(p, d^*, q).$$

We have[8]

Lemma 1. *If the positive real numbers* $\{a_n\}$ *in* (18) *and the system input* $\{u_n\}$ *(not necessarily given by* (9)*) satisfy the following conditions*

$$a_n \xrightarrow[n\to\infty]{} \infty \text{ and } a_n/\lambda_{\min}^{(p,q)}(n) \xrightarrow[n\to\infty]{} 0, \text{ a. s.} \tag{24}$$

for $(p, q) = (p_0, q^*)$ *and* $(p, q) = (p^*, q_0)$, *then* (p_n, q_n) *and* d_n *given by* (19) *and* (20) *respectively are consistent, i. e.*

$$(p_n, d_n, q_n) \xrightarrow[n\to\infty]{} (p_0, d_0, q_0), \text{ a. s.} \tag{25}$$

Proof of Theorem 1.

By Lemma 1 for proving (22), we only need to show that $\{a_n\}$ satisfying (16), satisfies (24) as well. For this it suffices to prove that

$$\liminf_{n\to\infty} n^{-1+(1+t_0)(\varepsilon+\delta)}\lambda_{\min}^{(p,q)}(n) \neq 0, \text{ a. s.} \tag{26}$$

for $(p, q) = (p_0, q^*)$ and $(p, q) = (p^*, q_0)$.

If (26) were not true, then there would exist a vector[7]

$$\eta = [\alpha^{(0)\tau} \cdots \alpha^{(p-1)\tau}\beta^{(d^*)\tau} \cdots \beta^{(q-1)\tau}]^\tau, \quad \|\eta\| = 1$$

satisfying

$$0 = \sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i (Adj A(z)) B(z) + \sum_{i=d^*}^{q-1} \beta^{(i)\tau} z^i \det A(z) I. \tag{27}$$

If $(p, q) = (p_0, q^*)$, then we have

$$\deg \left(\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i \right) \leqslant p_0 - 1. \tag{28}$$

If $(p, q) = (p^*, q_0)$, then by (27) we have

$$\deg \left(\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i \right) + (m-1)p_0 + q_0$$

$$= \deg \left(\sum_{i=d^*}^{q_0-1} \beta^{(i)\tau} z^i (\det A(z)) \right) \leqslant q_0 - 1 + mp_0$$

and hence

$$\deg \left(\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i \right) \leqslant p_0 - 1.$$

Thus (28) holds for both cases $(p, q) = (p_0, q^*)$ and $(p, q) = (p^*, q_0)$.

By Condition H_2 there are polynomial matrixes $M(z)$ and $N(z)$ so that

$$A(z)M(z) + B(z)N(z) = I.$$

Then from (27) we have

$$\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i Adj A(z) = \sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i Adj A(z)(A(z)M(z) + B(z)N(z))$$

$$= (\det A(z)) \left(\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i M(z) - \sum_{i=d^*}^{q-1} \beta^{(i)\tau} z^i N(z) \right). \tag{29}$$

From this and (28) we find that

$$\deg \left(\sum_{i=0}^{p-1} \alpha^{(i)\tau} z^i Adj A(z) \right) \leqslant p_0 - 1 + (m-1)p_0 = mp_0 - 1$$

$$< mp_0 = \deg(\det A(z)).$$

Consequently, from (29) we know $\alpha^{(i)} = 0$ for $0 \leqslant i \leqslant p-1$ and hence $\beta^{(i)} = 0$ for $d^* \leqslant i \leqslant q-1$ by (27). This contradicts $\|\eta\| = 1$ and thus we have verified (26).

Since M and $\overline{M}$ are finite sets, (22) means that $(p_n, d_n, q_n) = (p_0, d_0, q_0)$ for all sufficiently large n.

From (12) it is easy to see that

$$\|\theta(p, d^*, q) - \theta_n(p, d^*, q)\| = O(1/\lambda_{\min}^{(p_0, q_0)}(n)), \text{ a. s.}$$

which together with (26) implies (23).

IV. Identification of Continuous-Time Systems

We now consider the system described by (2)—(4) and still need Conditions H_1 and H_2.

The least squares estimate

$$\theta_t^\tau(p, d, q) = [-A_{1t}\cdots - A_{pt}B_{dt}\cdots B_{qt}] \tag{30}$$

for $\theta^\tau(p, d, q)$ defined by (10) is now given by

$$\theta_t(p, d, q) = \left(\int_0^t \varphi_s(p, d, q)\varphi_s^\tau(p, d, q)ds + I\right)^{-1}\int_0^t \varphi_s(p, d, q)y_s^\tau ds, \tag{31}$$

where

$$\varphi_t(p, d, q) = [sy_t^\tau\cdots s^p y_t^\tau s^d u_t^\tau s^{d+1}u_t^\tau\cdots s^q u_t^\tau]^\tau. \tag{32}$$

Similar to the discrete-time case, the estimates (p_t, q_t) for (p, q) and d_t for d_0 are respectively derived from

$$(p_t, q_t) = \arg\min_{(p,q)\in M} CIC_1(p, q)_t, \tag{33}$$

$$d_t = \arg\min_{d\in \bar{M}} CIC_2(d)_t, \tag{34}$$

where

$$CIC_1(p, q)_t = \sigma_t(p, d^*, q) + (p+q)a_t, \tag{35}$$

$$CIC_2(d)_t = \sigma_t(p_t, d, q_t) - d\cdot a_t, \tag{36}$$

$$\sigma_t(p, d, q) = \int_0^t \|y_s - \theta_t^\tau(p, d, q)\varphi_s(p, d, q\|^2 ds, \tag{37}$$

and a_t is any positive real function such that

$$a_t \xrightarrow[t\to\infty]{} \infty \text{ and } a_t/(t+1)^{1-2\varepsilon} \xrightarrow[t\to\infty]{} 0 \tag{38}$$

for some $\varepsilon \in \left(0, \dfrac{1}{2}\right)$.

Let $(w_t, \mathfrak{F}_t)$ be an l-dimensional Wiener process and $G(s) = 1 + g_1 s + \cdots + g_\mu s^\mu$, $\mu > q^*$ be an arbitrary polynomial with all roots being real and negative.

The excitation input u_t for the system (2)—(4) is defined as the solution of the following equation:

$$G(s)u_t = (1+t)^{-\varepsilon}w_t. \tag{39}$$

Here u_t corresponds to v_n in (9) for discrete-time systems and serves purely for identifying unknown parameters. For controlling the system while identifying its parameters, it is natural to require that the excitation source superimposed on the desired control tends to zero as time goes on. Now let us show this. In fact, we prove

Lemma 2. $U_t = [u_t^\tau\, su_t^\tau\cdots s^{\mu-1}u_t^\tau]^\tau \xrightarrow[t\to\infty]{} 0$, a. s. *and hence* $B(s)u_t \xrightarrow[t\to\infty]{} 0$, *where u_t is defined from* (39).

Proof. Set

$$F_g = \begin{bmatrix} -g_1 I & -g_2 I & \cdots\cdots & -g_\mu I \\ I & 0 & \cdots\cdots & 0 \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 \quad I & 0 \end{bmatrix}, \quad I = I_{l\times l}$$

and let $PF_gP^{-1} = J_g$ be the Jordan form with diagonal elements $-\lambda_i$, where $\lambda_i > 0$ since eigenvalues of F_g are the reciprocals of the roots of $G(s)$.

Clearly,

$$dPU_t = J_g PU_t dt + Pd[(1+t)^{-\varepsilon} w_t^\tau \ 0\cdots 0]^\tau, \tag{40}$$

and for proving $U_t \xrightarrow[t\to\infty]{} 0$, a. s. it suffices to show $PU_t \xrightarrow[t\to\infty]{} 0$.

Considering the equation for each component of PU_t, from (40) we see that for verifying $PU_t \xrightarrow[t\to\infty]{} 0$ we only need to show that $x_t \xrightarrow[t\to\infty]{} 0$, where x_t is the solution of the following equation:

$$dx_t = f_t dt - \lambda_i x_t dt + c^\tau d((1+t)^{-\varepsilon} w_t), \quad x_0 = 0, \tag{41}$$

where f_t is an $\mathfrak{F}_t$-adapted continuous process either tending to zero as $t \to \infty$ or identically equal to zero and c is a constant vector, which, in fact, is the first l elements of a row of P.

We have

$$x_t = \int_0^t e^{-\lambda_i(t-\lambda)} f_\lambda d\lambda - \varepsilon c^\tau \int_0^t e^{-\lambda_i(t-\lambda)}(1+\lambda)^{-1-\varepsilon} w_\lambda d\lambda$$

$$+ c^\tau \int_0^t e^{-\lambda_i(t-\lambda)}(1+\lambda)^{-\varepsilon} dw_\lambda \triangleq I_1(t) + I_2(t) + I_3(t).$$

It is easy to see that $I_1(t) \xrightarrow[t\to\infty]{} 0$ and $I_2(t) \xrightarrow[t\to\infty]{} 0$ if we note $f_t \xrightarrow[t\to\infty]{} 0$ and $(1+t)^{-1-\varepsilon} w_t \xrightarrow[t\to\infty]{} 0$, where the latter is seen from the iterated logarithm law[11]

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2lt \text{ loglog } t}} \|w_t\| = 1. \tag{42}$$

Setting

$$\tau(t) = \inf\left\{\tau: \int_0^\tau e^{2\lambda_i\lambda}(1+\lambda)^{-2\varepsilon} d\lambda = t\right\},$$

we know[11]

$$\int_0^{\tau(t)} e^{\lambda_i\lambda}(1+\lambda)^{-\varepsilon} dw_\lambda$$

is an l-dimensional Wiener process.

Then by the iterated logarithm law (42), we find that

$$\frac{1}{\sqrt{2la(T)\log\log a(T)}}\left|\int_0^T e^{\lambda_i\lambda}(1+\lambda)^{-\varepsilon}dw_\lambda\right| = O(1),$$

where

$$a(t) = \int_0^t e^{2\lambda_i\lambda}(1+\lambda)^{-2\varepsilon}d\lambda.$$

Hence we have

$$\|I_3(t)\| \leqslant O(e^{-\lambda_i t}(2la(t)\log\log a(t))^{1/2}) \xrightarrow[t\to\infty]{} 0.$$

Theorem 2. *Assume that Conditions* H_1 *and* H_2 *hold and* u_t *is given by* (39). *Then*

$$(p_t, d_t, q_t) \xrightarrow[t\to\infty]{} (p_0, d_0, q_0) \tag{43}$$

and

$$\|\theta_t(p_t\vee p_0,\ d_t\wedge d_0,\ q_t\vee q_0) - \theta(p_t\vee p_0,\ d_t\wedge d_0,\ q_t\vee q_0)\|$$
$$= O((1+t)^{-(1-2\varepsilon)}),\ \text{a. s.} \tag{44}$$

Before proving the theorem, we first prove two lemmas.

Lemma 3. *If the positive real function* a_t *in* (35), (36) *and the input* u_t *(not necessarily given by* (39)) *satisfying conditions*

$$a_t \xrightarrow[t\to\infty]{} \infty \text{ and } a_t/\lambda_{\min}^{(p,q)}(t) \xrightarrow[t\to\infty]{} 0 \text{ a. s.} \tag{45}$$

for $(p, q) = (p_0, q^*)$ *and* $(p, q) = (p^*, q_0)$, *then* (p_t, d_t, q_t) *given by* (33) *and* (34) *are strongly consistent, and*

$$\|\theta_t(p_t\vee p_0,\ d_t\wedge d_0,\ q_t\vee q_0) - \theta(p_t\vee p_0,\ d_t\wedge\ d_0,\ q_t\vee q_0)\|$$
$$= O(1/\lambda_{\min}^{(p_0,q_0)}(t)), \tag{46}$$

where $\lambda_{\min}^{(p,q)}(t)$ *denotes the smallest eigenvalue of*

$$\int_0^t \varphi_s(p,\ d^*,\ q)\varphi_s^\tau(p,\ d^*,\ q)ds.$$

Proof. The proof for strong consistency is a duplicate of those for Lemma 1 if we replace n by t there and the summation $\sum_{i=0}^{n-1}$ by integral $\int_0^t$, and note that

$$y_t = \theta^\tau(p^*, d^*, q^*)\varphi_t(p^*, d^*, q^*) = \theta^\tau(p_0, d_0, q_0)\varphi_t(p_0, d_0, q_0)$$

with $A_i = 0$ for $i > p_0$ and $B_j = 0$ for $j < d_0$ or $j > q_0$ as agreed in Lemma 1

The convergence rate (46) is a direct consequence of the following expression:

$$\tilde{\theta}_t(p, d^*, q) \triangleq \theta(p, d^*, q) - \theta_t(p, d^*, q)$$

$$= \left(\int_0^t \varphi_s(p, d^*, q)\varphi_s^\tau(p, d^*, q)ds + I\right)^{-1} \theta(p, d^*, q)$$

for $p \geq p_0$ *and* $q \geq q_0$.

Take an arbitrary polynomial of order $\nu = mp^* + q^* + 1$:

$$H(s) = 1 + h_1 s + \cdots + h_\nu s^\nu$$

with all roots being real and negative and define l-dimensional v_t from the equation

$$H(s)v_t = u_t \text{ or } E(s)v_t = (1+t)^{-\varepsilon}w_t, \tag{47}$$

where u_t is given by (39) and $E(s) = G(s)H(s) = 1 + e_1 s + \cdots + e_{\mu+\nu}s^{\mu+\nu}$.

Set

$$F_e = \begin{bmatrix} -e_1 I & -e_2 I \cdots\cdots & -e_{\mu+\nu}I \\ I & 0 \cdots\cdots\cdots & 0 \\ 0 & I \ddots & 0 \\ \vdots & \ddots \ddots & \vdots \\ 0 & \cdots\cdots 0 \quad I & 0 \end{bmatrix}, \quad I = I_{l\times l}, \tag{48}$$

and similarly to (40) we have

$$V_t = F_e \int_0^t e^{F_e(t-\lambda)}[(1+\lambda)^{-\varepsilon}w_\lambda^\tau 0\cdots 0]^\tau d\lambda + [(1+t)^{-\varepsilon}w_t^\tau 0\cdots 0]^\tau, \tag{49}$$

or equivalently,

$$dV_t = F_e V_t dt + [d((1+t)^{-\varepsilon}w_t^\tau)0\cdots 0]^\tau, \tag{50}$$

where $V_t = [v_t^\tau s v_t^\tau \cdots s^{\mu+\nu-1}v_t^\tau]^\tau$.

Lemma 4. *If V_t is defined by (49) and F_e is stable, then*

$$\frac{1}{T^{1-2\varepsilon}}\int_0^T V_t V_t^\tau dt \xrightarrow[t\to\infty]{} \frac{1}{1-2\varepsilon}R, \text{ a. s.}, \tag{51}$$

where R is positively definite and is expressed by

$$R = \int_0^\infty e^{F_e\lambda}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{F_e^\tau \lambda}d\lambda.$$

Proof. F_e is a stable matrix since $E(s)$ is stable. Then there is a positive definite matrix $\overline{P}$ $\left(\text{in fact, } \overline{P} = \int_0^\infty e^{F_e^\tau s}\cdot e^{F_e s}ds\right)$ so that

$$\overline{P}F_e + F_e^\tau \overline{P} = -I. \tag{52}$$

By using (52) and Ito's formula, we have

$$\begin{aligned} dV_t\overline{P}V_t = &- \|V_t\|^2 dt + (1+t)^{-2\varepsilon}\text{tr}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\overline{P}dt \\ &+ 2(1+t)^{-\varepsilon}V_t^\tau \overline{P}[I0\cdots 0]^\tau dw_t \\ &- 2\varepsilon(1+t)^{-1-\varepsilon}\cdot V_t\overline{P}[I0\cdots 0]^\tau w_t dt. \end{aligned} \tag{53}$$

Noticing[9,10]

$$\int_0^t (1+t)^{-\varepsilon} V_\lambda^\tau \bar{P}[I0\cdots 0]^\tau dw_\lambda = O(1) + o\left(\left(\int_0^t \|V_\lambda\|^2 d\lambda\right)^{1/2+\eta}\right), \quad \forall \eta > 0 \tag{54}$$

and

$$\int_0^t (1+t)^{-1-\varepsilon} V_\lambda^\tau \bar{P}[I0\cdots 0]^\tau w_\lambda d\lambda$$
$$= o\left(\int_0^t (1+\lambda)^{-1/2-1/2\varepsilon} \|V_\lambda\| d\lambda\right) = o\left(\left(\int_0^t \|V_\lambda\|^2 d\lambda\right)^{1/2}\right)$$

for which the iterated logarithm law (42) is invoked, from (53) we find that

$$\int_0^t \|V_\lambda\|^2 d\lambda = O((1+t)^{1-2\varepsilon}). \tag{55}$$

By stability of F_e we have the estimate

$$e^{F_e t} = O(e^{-\rho t}) \text{ for some } \rho > 0. \tag{56}$$

Using (55) and (56) leads to

$$\int_0^t e^{F_e(t-\lambda)} \int_0^\lambda (1+s)^{-1-\varepsilon} V_s w_s^\tau ds \cdot e^{F_e^\tau(t-\lambda)} d\lambda$$
$$= O\left(\int_0^t e^{-2\rho(t-\lambda)} \int_0^\lambda (1+s)^{-1/2-1/2\varepsilon} \|V_s\| ds \cdot d\lambda\right)$$
$$= O\left(\int_0^t e^{-2\rho(t-\lambda)} \left(\int_0^\lambda \|V_s\|^2 ds\right)^{1/2} d\lambda\right) = O((1+t)^{1/2-\varepsilon}) \tag{57}$$

and

$$\int_0^t e^{F_e(t-\lambda)} \int_0^\lambda (1+s)^{-\varepsilon} V_s dw_s^\tau \cdot e^{F_e^\tau(t-\lambda)} d\lambda$$
$$= O\left(\int_0^t e^{-2\rho(t-\lambda)} \left(\int_0^\lambda (1+s)^{-2\varepsilon} d\int_0^s \|V_\mu\|^2 d\mu \cdot ds\right)^{1/2+\eta} d\lambda\right)$$
$$= O\left(\int_0^t e^{-2\rho(t-\lambda)} (1+\lambda)^{(1-4\varepsilon)(1/2+\eta)} d\lambda\right)$$
$$= O((1+t)^{(1-4\varepsilon)(1/2+\eta)}). \tag{58}$$

Applying Ito's formula to $dV_tV_t^\tau$ and then using estimates (57) and (58), we find

$$\lim_{t\to\infty} \frac{1}{(1+t)^{1-2\varepsilon}} \int_0^t V_s V_s^\tau ds$$
$$= \lim_{t\to\infty} \frac{1}{(1+t)^{1-2\varepsilon}} \left\{-\varepsilon \int_0^t e^{F_e(t-\lambda)} \int_0^\lambda (1+s)^{-1-\varepsilon} (V_s w_s^\tau [I0\cdots 0]\right.$$
$$+ [I0\cdots 0]^\tau w_s \cdot V_s^\tau) ds \cdot e^{F_e^\tau(t-\lambda)} d\lambda$$
$$+ \int_0^t e^{F_e(t-\lambda)} \int_0^\lambda (1+s)^{-\varepsilon} ([I0\cdots 0]^\tau \cdot (dw_s) V_s^\tau + V_s dw_s^\tau [I0\cdots 0])$$

$$\cdot e^{F_e^\tau(t-\lambda)} d\lambda + \frac{1}{1-2\varepsilon}\int_0^t e^{F_e(t-\lambda)}\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}((1+\lambda)^{1-2\varepsilon}-1)e^{F_e^\tau(t-\lambda)} d\lambda\Big\}$$

$$= \frac{1}{1-2\varepsilon}\lim_{t\to\infty}\frac{1}{(1+t)^{1-2\varepsilon}}\int_0^t e^{F_e(t-\lambda)}\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}(1+\lambda)^{1-2\varepsilon}e^{F_e^\tau(t-\lambda)} d\lambda,$$

which obviously coincides with $\frac{1}{1-2\varepsilon}\cdot R$.

Since $\left\{F_e, \begin{bmatrix} I_l \\ 0 \\ \vdots \\ 0\end{bmatrix}\right\}$ is controllable, R clearly is positively definite.

Proof of Theorem 2. By Lemma 2 it suffices to show

$$\liminf_{t\to\infty}(1+t)^{-(1-2\varepsilon)}\cdot\lambda_{\min}^{(p,q)}(t)\neq 0, \text{ a. s.} \tag{59}$$

for $(p, q) = (p_0, q^*)$ and $(p, q) = (p^*, q_0)$.

Define vector

$$\psi_t(p, q) = [s(\mathrm{Adj}A(s))B(s)v_t^\tau\cdots s^p(\mathrm{Adj}A(s))B(s)v_t^\tau \cdot s^{d^*}\det A(s)v_t^\tau\cdots s^q\det A(s)v_t^\tau]^\tau, \tag{60}$$

which clearly satisfies the following equation

$$H(s)\psi_t(p, q) = (\det A(s))\varphi_t(p, d^*, q).$$

Since all roots of $H(s)$ are negative and $(\det A(s))/H(s)$ is a proper rational fraction, we have[9]

$$\lambda_{\min}\left(\int_0^t \psi_s(p, q)\psi_s^\tau(p, q)ds\right) = \inf_{\|x\|=1}\left(\int_0^t\left(\frac{\det A(s)}{H(s)}x^\tau\varphi_s(p, d^*, q)\right)\right)^2 ds$$

$$\leqslant c\cdot\inf_{\|x\|=1}\left(\int_0^t(x^\tau\varphi_s(p,d^*,q))^2ds\right) = c\cdot\lambda_{\min}^{(p,q)}(t), \tag{61}$$

where $c > 0$ is some constant and $\lambda_{\min}(X)$ denotes the smallest eigenvalue of a matrix X.

Hence for (59) it is sufficient to prove that

$$\liminf_{t\to\infty}(1+t)^{-(1-2\varepsilon)}\lambda_{\min}\left(\int_0^t\psi_s(p, q)\psi_s^\tau(p, q)ds\right)\neq 0, \text{ a. s..} \tag{62}$$

If (62) were not true, then there would exist a sequence of unit vectors $\{\eta_{t_k}\}$

$$\eta_{t_k} = [\alpha_{t_k}^{(0)\tau}\cdots\alpha_{t_k}^{(p-1)\tau}\beta_{t_k}^{(d^*)\tau}\cdots\beta_{t_k}^{(q-1)\tau}]^\tau \tag{63}$$

so that

$$\lim_{k\to\infty}(1+t_k)^{-(1-2\varepsilon)}\left(\int_0^{t_k}(\eta_{t_k}^\tau\psi_s(p, q))^2ds\right) = 0, \tag{64}$$

where $\alpha_{t_k}^{(i)}(0\leqslant i\leqslant p-1)$ and $\beta_{t_k}^{(j)}(d^*\leqslant j\leqslant q-1)$ are m- and l-dimensional, respectively.

By (60) we see that

$$\eta_{t_k}^{\tau}\phi_t(p, q) = s\sum_{i=0}^{p-1} \alpha_{t_k}^{(i)^\tau} s^i(\mathrm{Adj}A(s))B(s)v_t + s\sum_{i=d^*}^{q-1} \beta_{t_k}^{(i)^\tau} s^i \det A(s)v_t$$

$$\triangleq [h_{t_k}^{(0)^\tau} h_{t_k}^{(1)^\tau}\cdots h_{t_k}^{(\mu+\nu-1)^\tau}]v_t, \tag{65}$$

where $h_{t_k}^{(i)}$ is l-dimensional and bounded in k. Then from (64) and Lemma 3 it follows that

$$0 = \lim_{k\to\infty} (1+t_k)^{-(1-2\varepsilon)} [h_{t_k}^{(0)^\tau}\cdots h_{t_k}^{(\mu+\nu-1)^\tau}] \int_0^{t_k} V_s V_s^\tau ds [h_{t_k}^{(0)^\tau}\cdots h_{t_k}^{(\mu+\nu-1)^\tau}]^\tau$$

$$\geqslant \limsup_{k\to\infty} [h_{t_k}^{(0)^\tau}\cdots h_{t_k}^{(\mu+\nu-1)^\tau}] \frac{1}{2(1-2\varepsilon)} R[h_{t_k}^{(0)^\tau}\cdots h_{t_k}^{(\mu+\nu-1)^\tau}]^\tau.$$

Hence there holds $\lim_{k\to\infty} h_{t_k}^{(i)} = 0$, $0\leqslant i\leqslant \mu+\nu-1$ and for any complex s we have

$$\lim_{k\to\infty} H_{t_k}(s) \triangleq \lim_{k\to\infty} s\sum_{i=0}^{p-1} \alpha_{t_k}^{(i)^\tau} s^i(\mathrm{Adj}\ A(s))B(s)$$

$$+ s\sum_{i=d^*}^{q-1} \beta_{t_k}^{(i)^\tau} s^i(\det A(s))I = 0. \tag{66}$$

Let

$$\eta = [\alpha^{(0)^\tau}\cdots\alpha^{(p-1)^\tau}\ \beta^{(d*)^\tau}\cdots\beta^{(q-1)^\tau}]^\tau \text{ with } \|\eta\| = 1$$

be a limit point of $\{\eta_{t_k}\}$.

Then (66) yields

$$\sum_{i=0}^{p-1} \alpha^{(i)^\tau} s^i(\mathrm{Adj}A(s))B(s) + \sum_{i=d*}^{q-1} \beta^{(i)^\tau} s^i(\det A(s))I = 0,$$

which is exactly the same as (27). Its impossibility is proved in Theorem 1. The contradiction proves the validity of (62) and at the same time completes the proof.

V. Conclusion

This paper concerns the experiment design for identifying linear input-output system. The characteristics of the proposed methods consist of the following: (i) Unknown coefficients, system orders and the time delay all are consistently estimated. (ii) Neither stability nor minimum-phase of the system is imposed on the system. (iii) Both discrete-time and continuous-time systems have similarly been analyzed. To authors' knowledge, on system identification this is the first work without requiring any condition on the system behavior.

Finally, we would like to draw reader's attention to some open problems: It is desirable to remove restriction that the upper bounds p^* and q^* and the lower bound d^* are known; it is of interest to develop adaptive control theory without imposing stability and minimum phase on the system.

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